Mathematical Physics: Individual All-Round Competition

To complete this assignment, answer 1 out of the 2 problems.

__Problem 1__

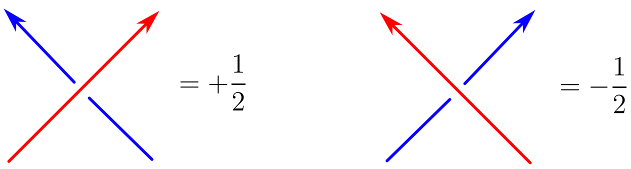
The linking number. A wire along an oriented loop $L_1 \subset \mathbb{R}^3$ in the three-dimensional space carries a constant electric current I. Let L_2 be another loop that does not intersect L_1.

1. We have
$$\int_{L_2} \vec{B} \cdot d\vec{l}_2 = \mathrm{lk}(L_1, L_2) \cdot \mu_0 I.$$
Explain why geometrically the quantity $\mathrm{lk}(L_1, L_2)$ can be referred to as the "linking number" of L_1 and L_2 by showing that, when it is non-zero, one can not separate L_1 and L_2. Also argue that it remains constant under deformations of L_1 and L_2 as long as they don't cross each other.

2. Give a physics derivation of the Gauss linking integral formula
$$\mathrm{lk}(L_1, L_2) = \frac{1}{4\pi} \int_{L_1 \times L_2} d\vec{l}_2 \cdot d\vec{l}_1 \times \frac{\vec{l}_1 - \vec{l}_2}{|\vec{l}_1 - \vec{l}_2|^3}. \tag{0.1}$$
Here $\vec{l}_{1,2}$ respectively parametrize $L_{1,2}$, while "$\times$" and "$\cdot$" are respectively the cross and scalar products in $\mathbb{R}^3$.

3. Verify that the integral (0.1) is symmetric under $L_1 \leftrightarrow L_2$. Explain this physically using electric-magnetic duality. (Hint: consider a "magnetic current" I_m along L_2. Also you don't need to keep track of constants such as μ_0, ϵ_0 and c.)

4. Argue that given a projection of the link, the linking number can be computed by summing over crossings with each contributing $\pm\frac{1}{2}$ according to the following rule.



Here, the two strands belong to the two different components, but it doesn't matter which is which due to the symmetry we have seen. (Hint: arrange the configuration of L_1 and L_2 so that you can evaluate/approximate the integral. You may find the following formula useful,
$$\int_{-\infty}^{\infty} dx \frac{r^2}{(x^2 + r^2)^{\frac{3}{2}}} = 2.)$$

__Problem 2__

A non-relativistic quantum particle of spin zero moves on the real line $\mathbb{R}$ subjected to the linear potential
$$V(x) = 2gx$$
(x being the standard Cartesian coordinate in $\mathbb{R}$ and g a real parameter).

Find the heat kernel $\langle x' | e^{-\beta H} | x \rangle$ of this quantum system by a direct computation of the path integral.